Impawn Rate Optimisation in Inventory Financing: A Canonical Vine Copula-based Approach

Abstract

In the inventory financing business, an optimal impawn rate (loan-to-value ratio) can help the inventory financing providers (IFPs, she) maintain competitiveness in the inventory financing market. However, the literature has been silent on how IFPs can optimise the business through the optimisation of the impawn rate. This study examines the role of the optimal impawn rate in the inventory financing business. The key to setting the optimal impawn rate is first evaluating default probability and then incorporating this into the profit function. We use a data-driven approach to explore the copula model in setting the optimal impawn rate. Through numerical analysis, we find that the Clayton canonical vine copula has a better performance for the prediction of default probability than the multivariate normal distribution (MVN) and can thus be used to evaluate default probability. In addition, we uncover that setting multiple impawn rates for different collaterals allows inventory financing to yield a higher profit. Further, although the interest rate, industrial impawn rate, and optimal impawn rate have strong effects on inventory financing profit, interestingly, the relationship between them is marginally diminishing.

Keywords: Impawn Rate Optimisation; Inventory Financing; Inventory Financing Provider; Canonical Vine Copula

1. Introduction

1.1. Background and Research Questions

In inventory financing, inventory is used as a pledge for the purpose of risk aversion. Compared with conventional financing, inventory financing plays an essential role in alleviating the capital constraints for small and medium-size enterprises (SMEs) by allowing them to improve cash flow and the ability to fulfil customer orders (Buzacott & Zhang, 2004). Traditionally, inventory financing is often provided by banks and other financial institutions, such as Commercial Capital LLC and Crossroads Financial. Interestingly, more firms, including third-party logistics providers (TPLs), are engaging in inventory financing service (Capgemini, 2016; Mayer, 2013) as the profit margin generated from traditional logistics operations decreases (Hofmann, 2009; Liu & Zhou, 2017; Li & Chen, 2018). For instance, UPS founded UPS Capital to provide in-transit inventory financing services, and Schneider Logistics Inc. collaborated with U.S. banks to provide better financial solutions to its capital-constrained clients [1].

Because pledged inventory is not guaranteed to maintain its initial value (He et al., 2012, 2014; Wu et al., 2019), when providing inventory financing services, an IFP sets an impawn rate to control the financing risk. The impawn rate is the ratio between the loan ascribed to the collateral and the market value of the collateral. A high impawn rate means the borrower (he) could receive more money based on his collateral, and a low impawn rate indicates that he could receive less money. Currently, the settlement of the impawn rate is diverse and based on the industrial experience of IFPs. For example, Commercial Capital LLC advances up to 80% of the appraised value of collateral. UPS Capital claims that the impawn rate can be up to 100%. Obviously, an impawn rate determined in this way does not reveal the fluctuating value of inventory, as the market is dynamic and the corresponding likelihood that the customer will default changes in different funding cycles. To illustrate, from 9/30/2008 to 11/28/2008, the price of aluminium alloy dropped approximately 36.38% (from $ 2,130 /ton to $ 1,355 /ton). From 5/31/2018 to 7/31/2018, the price only dropped approximately 4.5% in three months (from $ 1,855 /ton to $ 1,775 /ton) [2]. If an IFP sets an optimal impawn rate from 9/30/2008 to 11/28/2008 based on previous experience, she will face a serious risk of default by customers who use aluminium alloy as a collateral, since they may not be able to return the money at the end of the funding cycle. In contrast, if the IFP allocates a low impawn rate to aluminium alloy from 5/31/2018 to 7/31/2018, although the default risk is lower, the low rate would not increase competitiveness in the inventory financing market. Therefore, evaluating

[1] UPS Capital: `https://upscapital.com/`; Schneider Logistics: `https://schneider.com/`

[2] London Metal Exchange (LME): `https://www.lme.com/`

the default probability based on changing collateral value and setting the corresponding optimal impawn rate would be helpful to optimise the inventory financing business.

Moreover, impawn rate optimisation has been extensively investigated, and many beneficial explorations on the volatility and risk management of pledges have already been made (Ni et al., 2016; Zhang et al., 2017; Wang et al., 2018). However, a majority of the literature assumes that the default probability in financial services has no relationship with the changing value of collateral and can be set as a fixed parameter. In reality, the default motivation of the borrower is not static and is strongly linked with the fluctuating value of collateral (He et al., 2012, 2014). Motivated by the practical and theoretical cases above, the focus of our research is thus on how IFPs dynamically evaluate the default probability based on the changing value of collateral and set a corresponding impawn rate to optimise their inventory financing business. Specifically, we examine the following research questions. *(1) How does an IFP effectively evaluate default probability in the inventory financing business? (2) How can the default probability be incorporated into the objective profit function to set the optimal impawn rate for inventory financing? (3) How do other factors, including the willingness to take risks, the liquidity risk of collateral, the interest rate, and the industrial impawn rate, affect how an IFP sets the optimal impawn rate?* In the next section, we briefly discuss how we address these questions and outline our contributions.

1.2. Main Findings and Contributions

To address the research questions above, we construct an objective profit function consisting of demand for money, default probability, interest, and the supervision cost of collateral, showing the relationship between the impawn rate and the profit that an IFP can gain from the inventory financing business. By iteratively evaluating the default probability in the objective profit function, we can dynamically calculate the optimal impawn rate for each funding cycle. The following demonstrates our approach to each of the research questions and our contributions.

First, we provide a new data-driven approach to set the optimal impawn rate in inventory financing, which has not been observed in the existing inventory financing literature (Buzacott & Zhang, 2004; Wang et al., 2018; Zhang et al., 2016). To manage the default risk caused by fluctuating collateral prices, this study investigates how an IFP dynamically adjusts the impawn rate in inventory financing and varies from the existing literature in that the impawn rate is the given parameter (Boissel et al., 2017; Buzacott & Zhang, 2004). To calculate the optimal impawn rate, we first construct the objective profit function of the inventory financing business, which comprises the function of funding demand, the function of default probability, and the revenue from inventory financing. Based on the returns generated by the predictive models for each funding cycle, we dynamically modify the parameters of the distribution function and derive the optimal impawn rate for inventory financing during each funding cycle.

Second, we identify an effective copula-based approach for the IFPs to dynamically evaluate the default probability for each funding cycle, which differs from the approaches in the existing literature that use the given parameters (e.g., Buzacott & Zhang (2004)) or the distribution function with unchanged parameters (e.g., exponential distribution, as mentioned by Wang et al. (2018)) to evaluate the default risk. To parameterise the distribution function, we first determine an effective model to predict future returns of collateral. In finance and economics, bivariate copulas have been widely applied due to their effectiveness in capturing the dependence structure between pair time series and forecasting returns for pair assets (Fan & Patton., 2014). By adopting the canonical vine copula, bivariate copulas can be further extended to multidimensional copulas to capture the dependence structure among multi-time series and predict returns for multiple assets (Aas et al., 2009). Based on the real data, we extend the bivariate Clayton copula to the Clayton canonical vine copula and compare its predictive performance with that of the multivariate normal distribution (MVN), a benchmark model that has been widely used to predict multidimensional returns in the financial field (Low et al., 2013). Through the comparative analysis, we reveal that the Clayton canonical vine copula can better capture the dependence structure of collateral returns than the MVN and thus help the distribution function more accurately to evaluate the default probability for each funding cycle.

Third, the insights derived from our analysis provide important managerial implications that can help IFPs further improve their inventory financing business. More specifically, our analysis reveals relationships among the critical factors (e.g., impawn rate, interest rate, liquidity risk, risk-taking ability, and industrial impawn rate) and their effects on the financial performance of IFPs. For instance, the marginal effect of the interest rate on the optimal impawn rate gradually decreases; when the interest rate is low, a one-unit increase in the interest rate can greatly motivate the IFP to set a higher impawn rate. However, with the increase of default risk, the motivation for the IFP to set a higher impawn rate decreases. Similarly, the impawn rate and industrial impawn rate have no strong linear relationship. However, the industrial impawn rate has a strong effect on the expected profit. Therefore, with a decrease of the industrial impawn rate, the IFP can gain more profit from her inventory financing business providing she can set the optimal impawn rate. Therefore, when the inventory financing market is depressed and all lenders are inclined to set a low impawn rate, it is easier for the IFP to gain more profit if she can accurately evaluate the default probability.

The remainder of the paper is organised in the following manner. Section 2 discusses prior research in related areas. Section 3 presents the setup of the model. Section 4 briefly describes the source data we use and shows the results for the optimal impawn rate. Section 5 further extends the original business model to consider the factor of the borrower in the settlement of the optimal impawn rate. Section 6 concludes the paper. The proof of the proposition is shown in Appendix A.

2. Related Literature

In inventory financing, the IFP can make full use of her control of collateral and set a dynamical impawn rate to maximise the profit for her inventory financing business. To specify how the IFP optimises her inventory financing business through the management of the impawn rate, the literature related to inventory financing and impawn rate optimisation from the perspective of operations management, finance, and economics is reviewed.

2.1. Inventory Financing

Since the 2008 financial crisis, SMEs in supply chains have faced constrained cash flows and found it difficult to obtain financing support from financial institutions due to a lack of fixed assets (Jia et al., 2020; Zhao & Huchzermeier, 2018). Against this background, IFPs provide inventory financing to SMEs to relieve financing constraints. In industrial supply chains, the operation of the whole chain is constructed through collaboration and coordination with the focal company, normally recognised as a manufacturer, and other chain members (Nag, 2014). To produce industrial goods, the manufacturer (focal company) obtains orders from the market and turns to suppliers, requesting that they provide various materials for manufacturing industrial products. As the volume of industrial goods is relatively large, suppliers need enormous capital to purchase various materials and equipment for manufacturing products. Suppliers, particularly SME suppliers, cannot afford capital for production by themselves, which causes the instability of the whole supply chain operation. To prevent breaks in production, the IFP provides an inventory financing service to capital-constrained suppliers. Raw materials such as copper, aluminium alloy, and lead, which belong to the supplier, can be used as collateral (Liu & Zhou, 2017).

Some studies have investigated inventory financing from a qualitative perspective. For example, Hofmann (2009) developed the concept of inventory financing, offering initial insights into the significance of the field. He demonstrated that the value and amount of goods have a strong effect on the profit yielded by the inventory financing business. Recently, Li & Chen (2018) adopted a multiple case study approach to identify how the IFP takes advantage of the financial service to generate sustainable competitive advantage. In contrast, some studies have explored how the

IFP plays a role in the financial service from the quantitative perspective. Most of the research has investigated inventory financing service from the perspective of banks. For example, Hwan Lee & Rhee (2010) studied how inventory financing costs affect supply chain coordination under four coordination mechanisms: an all-unit quantity discount, buyback, two-part tariffs, and revenue sharing. The authors demonstrated that positive inventory financing costs make revenue sharing less profitable than other mechanisms. Buzacott & Zhang (2004) constructed a multi-period inventory control model to investigate the interplay between inventory decisions and asset-based financing. They concluded that asset-based financing can help retailers improve profit. In addition, a number of studies have examined how TPLs provide financing service. For example, Chen & Cai (2011) developed an extended supply chain model with a supplier, a budget-constrained retailer, a bank, and a TPL, comparing different roles of the TPL in providing financial services. They identified that the whole supply chain performs better in the control role model in which the TPL integrates logistics and financial services. Although the above-mentioned studies on inventory financing business are significant and promising, they are silent regarding how the IFP manages the impawn rate to optimise her inventory financing business.

2.2. Impawn Rate Optimisation

In financial markets, the impawn rate (also called a haircut or margin requirement) refers to a reduction applied to the value of an asset (Ashcraft et al., 2011; He et al., 2012). The settlement of the impawn rate depends on several factors, including the risk (i.e. the volatility of its price) and liquidity (i.e. how easy it is to sell it quickly without a loss of value) of an asset type. An overly high impawn rate or low haircut exposes the lender to risk that is the result of the fluctuating value of collateral. For this reason, the impawn rate tends to decrease during a crisis due to liquidity issues (Brunnermeier & Pedersen, 2008; Trebesch & Zabel, 2017). This kind of negative correlation between the duration of default and the size of the impawn rate has already been empirically documented by Boissel et al. (2017), Gorton & Metrick (2012), Luo & Wang (2018) and Tobias et al. (2010). However, all of these studies have mainly investigated the mechanism of the impawn rate (haircut or margin requirement) by setting bonds or securities as collateral, rather than in the context in which the IFP sets raw materials as collateral. In actuality, the loan secured by collateral with an impawn rate (inventory financing) has played an important role in facilitating commercial activities in capital-constrained supply chains (He et al., 2012; Liu & Zhou, 2017).

For real-world financial institutions, the impawn rate is a key funding constraint. Lenders must consider what size buffer is sufficient to cover the risk of not being able to sell an asset at its current value. The reason that Bear Sterns, Lehman and AIG collapsed was because they were unable to meet their margin constraints (Ashcraft et al., 2011). To determine the impawn rate, another important factor that the lender needs to consider is the default rate. Some literature has already investigated the relationship between the impawn rate and borrower default. For example, Boissel et al. (2017) investigated the effect of haircut policy (an increase or decrease of the haircut) on the sensitivity of repo market rates to sovereign default risk during the Eurozone crisis of 2008-2012. The authors found that raising the haircut is ineffective when the sovereign default risk is extremely high, as in 2011. To study sovereign lending and default, Luo & Wang (2018) constructed a dynamic contracting model with private information, explaining the positive correlation between the size of haircut and the duration of default. Simultaneously, the evaluation of default probability has also been underlined by the operation management literature. For example, Shi & Zhang (2010) incorporated default risk into the trade credit offered by a supplier. Wang et al. (2018) described the probability that a retailer will pay on time with an exponential distribution function. Kouvelis & Zhao (2012) examined how trade credit risk affects operation decisions. However, all of these studies have evaluated default probability with a given parameter or a distribution function with fixed parameters, which isolates default risk from the fluctuating prices of pledged collateral.

To incorporate the fluctuating value of collateral in the evaluation of probability, the first step is to predict the trend of future collateral prices. In the finance literature, the MVN is a classical model that has been used to simulate asset returns. However, the model's accuracy and efficiency have been questioned because of its inability to capture the asymmetric structure of the time series (Low et al., 2013). Fortunately, this situation has changed since the concept of canonical vine copulas was raised by Aas et al. (2009). Using the pair-copula decomposition of a general multivariate distribution, the authors demonstrated that the canonical vine copula can take advantage of its flexibility to capture the dependency structure of the time series to accurately simulate future returns based on historical time series.

The impawn rate in the inventory financing field has been previously investigated. For example, using the formula AR(1)-GARCH(1,1)-GED and the value at risk (VaR) model, He et al. (2012) dynamically optimised the impawn rate of steel in different risk windows. By comparing the conditional value at risk (CVaR) and VaR model, He et al. (2014) further dynamically optimised the impawn rate of an inventory portfolio in various risk windows. However, the authors assumed that a changing impawn rate does not influence funding demand, which is inconsistent, in actuality, with the business world (Ashcraft et al., 2011). To better control risk due to the fluctuation of collateral prices and maintain the competitiveness of IFPs, it is necessary to dynamically optimise the impawn rate by considering both default probability and funding demand.

In summary, this research intends to fill the gap in the existing literature by dynamically parameterising the function of default probability in inventory financing for each funding cycle using canonical vine copulas. Based on the parameterised function of expected profit, the impawn rate can be dynamically optimised to maximise the profit of the inventory financing business.

3. Model

The IFP has n collaterals and provides corresponding funding for them (See Fig. 1). The funding demand for each collateral unit is $M_{i,0}$. θ_i is the impawn rate used by the IFP to manage the risk of each collateral unit. $\theta_i p_{i,0} q_{i,0} = M_{i,0}$. $p_{i,0}$ is the initial price of i^{th} collateral unit and $q_{i,0}$ is the initial quantity of i^{th} collateral unit. The funding demand for each collateral unit is influenced by the settlement of the impawn rate (i.e. $M_{i,0}(\theta_i) = \alpha_i + \beta_i(\theta_i - \bar{\theta})$). A similar linear model for the demand for money has been widely used in economics (Ashcraft et al., 2011; Christoffersen & Musto, 2002; Suntory & Disciplines, 2007). $M_{i,0}(0) = 0$ and $\beta_i > 0$, which means a higher impawn rate will attract more collateral, and there is no funding demand when the impawn rate is set as 0. The demand function, although quite general, is assumed to be linear in the impawn rate. The linearity assumed here does not affect our analysis in which primary interest focuses on the evaluation of default probability and how the optimal impawn rate depends on relevant factors. The interest rate is r, and the length of interval is k. $M_{i,j} = M_{i,0}\exp(kjr)$. When $M_{i,j} = \theta_i p_{i,j} q_{i,j-1}$, the IFP does not need to call the margin. When $M_{i,j} < \theta_i p_{i,j} q_{i,j-1}$, the borrower can take back extra collateral. When $M_{i,j} > \theta_i p_{i,j} q_{i,j-1}$, the borrower is required to bring more collateral to the IFP until $M_{i,j} = \theta_i p_{i,j} q_{i,j}$. Some risk does exist because if the borrower does not fulfil the contract, the IFP must take time to deal with the collateral. However, it is very likely that the difference between the initial market value and the value realised after liquidation is greater than 0. Therefore, when dealing with collateral, the loss suffered by the IFP is $M_{i,j} - (1-\rho_i)p_{i,j}q_{i,j-1}$, and ρ_i is the level of liquidity risk of i^{th} collateral unit. See Table 1 for a summary of the notations.

3.1. Objective Function of Expected Profit

Assuming that borrowers who provide the i^{th} collateral unit do not fulfil the contract, the loss suffered by the IFP on the i^{th} collateral unit at the end of the interval is as follows:

$$L_i = [M_{i,j} - p_{i,j}q_{i,j-1}(1-\rho_i)]\exp(-kjr) \tag{1}$$

Because $M_{i,j} = M_{i,0}\exp(kjr)$, Eq. (1) can thus be further transformed into Eq. (2).

$$L_i = M_{i,0} - p_{i,j}q_{i,j-1}(1-\rho_i)\exp(-kjr) \tag{2}$$

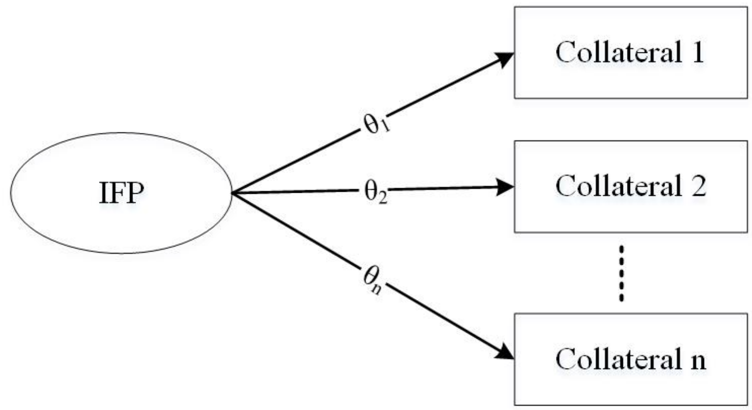


Fig. 1: Optimal Impawn Rate for Each Collateral Unit

Table 1: Summary of Notation and Assumptions

Symbol	Description	Assumption
k	The length of the interval	k is nonnegative integer
m	The number of intervals	m is nonnegative integer
n	The number of collateral units	n is nonnegative integer
$M_{i,0}$	The amount of the loan for the i^{th} collateral unit	$M_{i,0} \geq 0$
r	The interest rate provided by the IFP	$0 < r$
$p_{i,j}$	The price of i^{th} collateral at the end of j^{th} interval	$0 < p_{i,j}$
$q_{i,j}$	The quantity of i^{th} collateral at the end of j^{th} interval	$0 \leq q_{i,j}$
θ_i	The impawn rate (decision variable)	$0 < \theta_i < 1$
$\bar{\theta}$	The industrial impawn rate	$0 < \bar{\theta} < 1$
τ	The level of risk that the IFP is willing to undertake	$0 \leq \tau < 1$
ρ_i	The level of liquidity risk of i^{th} collateral unit	$0 \leq \rho_i \leq 1$
α_i	The constant of the correlation between $(\theta_i - \bar{\theta})$ and $M_{i,0}$	$0 < \alpha_i$
β_i	The coefficient of the correlation between $(\theta_i - \bar{\theta})$ and $M_{i,0}$	$0 < \beta_i$
g_i	The supervision fee for one unit of i^{th} collateral in each interval	$0 < g_i$

Assuming that the IFP can tolerate the loss $\bar{M}_{i,0}$, $\bar{M}_{i,0} = \tau M_{i,0}$ (τ is the level that the IFP can tolerate). We can further calculate the probability of loss for the i^{th} collateral unit. For the IFP, the probability of loss in the j^{th} interval is as follows:

$$\mathbb{P}(\bar{M}_{i,0} \leq L_i) = \mathbb{P}(\tau M_{i,0} \leq M_{i,0} - p_{i,j} q_{i,j-1}(1-\rho_i)\exp(-kjr)) \tag{3}$$

Because

$$q_{i,j-1} = \frac{M_{i,0}\exp(k(j-1)r)}{\theta_i p_{i,j-1}} \tag{4}$$

Eq. (3) can thus be further transformed into:

$$\mathbb{P}(\bar{M}_{i,0} \leq L_i) = \mathbb{P}(\tau \leq 1 - \frac{(1-\rho_i)\frac{p_{i,j}}{p_{i,j-1}}}{\theta_i \exp(kr)}) \tag{5}$$

and Eq. (5) can be further transformed into:

$$\mathbb{P}(\bar{M}_{i,0} \leq L_i) = \mathbb{P}(\frac{p_{i,j}}{p_{i,j-1}} \leq \frac{\theta_i(1-\tau)\exp(kr)}{1-\rho_i}) \tag{6}$$

We set $\frac{p_{i,j}}{p_{i,j-1}} = P_i$ and $\frac{\theta_i(1-\tau)\exp(kr)}{1-\rho_i} = Z$. Then:

$$\mathbb{P}(\bar{M}_{i,0} \leq L_i) = \mathbb{P}(P_i \leq Z) \tag{7}$$

If we add 'ln' to both the left and right term in the second part of Eq.(7), then we have:

$$\mathbb{P}(\bar{M}_{i,0} \leq L_i) = \mathbb{P}(\ln P_i \leq \ln Z) \tag{8}$$

$\ln P_i$ is logarithmic return of i^{th} collateral unit. Therefore, we have Proposition 1.

Proposition 1. *When the IFP can tolerate only the loss $\bar{M}_{i,0}$, $\bar{M}_{i,0} = \tau M_{i,0}$ (τ is the level of risk that the IFP can tolerate). The probability of loss for the i^{th} collateral unit in the j^{th} interval is $\mathbb{P}(\ln P_i \leq \ln Z)$. $P_i = \frac{p_{i,j}}{p_{i,j-1}}$ and $Z = \frac{\theta_i(1-\tau)\exp(kr)}{1-\rho_i}$.*

Proposition 1 describes the probability of loss for the i^{th} collateral unit, which is significantly affected by the impawn rate (θ_i) set by the IFP. With an increase in the impawn rate set by the IFP, the probability of loss would also increase.

When the IFP receives the collateral, she expends energy and takes the time to manage it for each funding cycle. Supervision costs also exist here, which are shown as follows:

$$G_i = \frac{M_{i,0}}{\theta_i p_{i0}} g_i m \tag{9}$$

Therefore, the profit that the borrower expects to earn for the i^{th} collateral unit is as follows:

$$\pi_i(\theta_i) = \sum_{j=1}^{m} [\alpha_i + \beta_i(\theta_i - \bar{\theta})][1 - \mathbb{P}(\ln P_i \leq \ln Z)][\exp(kjr) - \exp(k(j-1)r)] - G_i \tag{10}$$

$\mathbb{P}(\ln P_i \leq \ln Z)$ is the default probability. $\exp(kjr) - \exp(k(j-1)r)$ is the interest owned by the IFP in the j^{th} funding cycle. Therefore, the expected revenue on the i^{th} collateral unit for the j^{th} funding cycle is $[\alpha_i + \beta_i(\theta_i - \bar{\theta})][1 - \mathbb{P}(\ln P_i \leq \ln Z)][\exp(kjr) - \exp(k(j-1)r)]$. Eq. (10) can be further simplified into:

$$\pi_i(\theta_i) = [\alpha_i + \beta_i(\theta_i - \bar{\theta})][1 - \mathbb{P}(\ln P_i \leq \ln Z)][\exp(kjr) - 1] - G_i \tag{11}$$

To calculate the optimal impawn rate with Eq. (11), we can take its first-order and second-order derivative. Here, we have Proposition 2.

Proposition 2. *There exists a single optimal impawn rate θ_i^* in the interval $(0, \min(\exp(\delta + \mu - \omega), 1))$ that maximises the profit of the IFP on i^{th} collateral unit and $\omega = \ln \frac{1-\tau}{1-\rho_i} + kr$. (The proof is in Appendix A.1).*

In Proposition 2, μ and δ are the mean and variance of logarithmic returns of i^{th} collateral unit simulated by the predictive model. From the proof in Appendix A.1, we know that there exists an optimal impawn rate in the interval $(0, \min(\exp(\delta + \mu - \omega), 1))$ that maximises the profit of the IFP for the i^{th} collateral unit. Based on the calculated optimal impawn rate for each collateral unit, the total optimal profit from the inventory financing business is as follows:

$$\Pi(\theta_1^*, \theta_2^*, ..., \theta_n^*) = \sum_{i=1}^{n} [\pi_i(\theta_i^*) - G_i(\theta_i^*)] \tag{12}$$

3.2. Canonical Vine Copula

To optimise the impawn rate, the returns from collateral should first be simulated to parameterise the function of default probability. In this study, we adopt the canonical vine copula to simulate the returns from collateral, as it can capture the dependence structure among different time series very well (Aas et al., 2009).

Every cumulative joint distribution function (CDF) reveals the marginal behaviour of individual values and their dependency structure. However, the CDF can be expressed in another way. Consider a vector $\boldsymbol{X} = (X_1, ..., X_n)$ of random variables with a joint CDF $F(x_1, ..., x_n)$ and marginal distributions $F_i (i = 1, ..., n)$; there exists a copula to describe the dependence structure among the marginal distribution functions based on (Sklar, 1973) as follows:

$$F(x_1, ..., x_n) = C[F_1(x_1), ..., F_n(x_n)] \tag{13}$$

Using the transformation $F_i(X_i) = U_i$, the copula from Eq. (13) has the following expression:

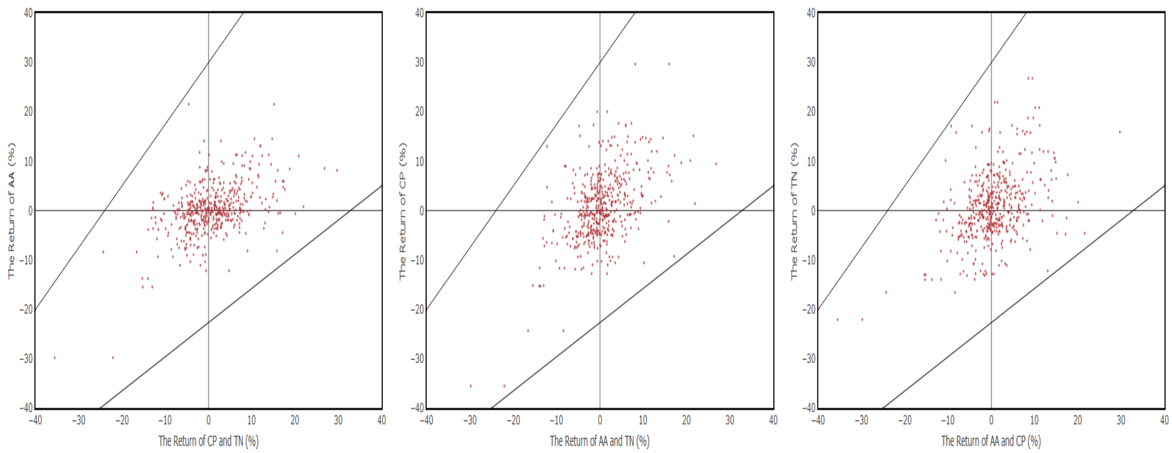
$$F(x_1, ..., x_n) = C[F_1(x_1), ..., F_n(x_n)] = C(u_1, ..., u_n) = \mathbb{P}(U_1 \leq u_1, ..., U_n \leq u_n) \tag{14}$$

where $C(u_1, ..., u_n)$ is a CDF for a multivariate vector with support in $[0, 1]^n$ and uniform margins. If we assume marginal CDF F_i and the copula function C in Eq.(14) to be differentiable, the joint density function $f(x_1, ..., x_n)$ and the density of the copula $c(u_1, ..., u_n)$ can be separately defined as:

$$f(x_1, ..., x_n) = c_{1,...,n}[F_1(x_1), ..., F_n(x_n)] \cdot f_1(x_1)...f_n(x_n) \tag{15}$$

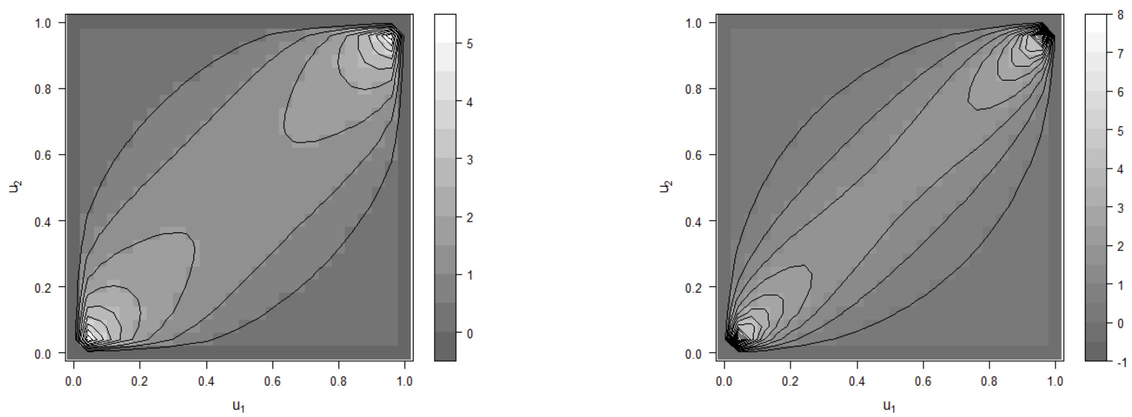
$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1, ..., \partial u_n} \tag{16}$$

Due to the structure of the lower tail of the time series (See Fig. 2), the Clayton copula can be used to capture the dependence structure of the value of collateral (Low et al., 2013). To illustrate, from Fig. 3 and Fig. 3d, we can see that compared with other copulas, the density of the Clayton copula has a similar structure as the time series of the sample collateral (See the two lines in each

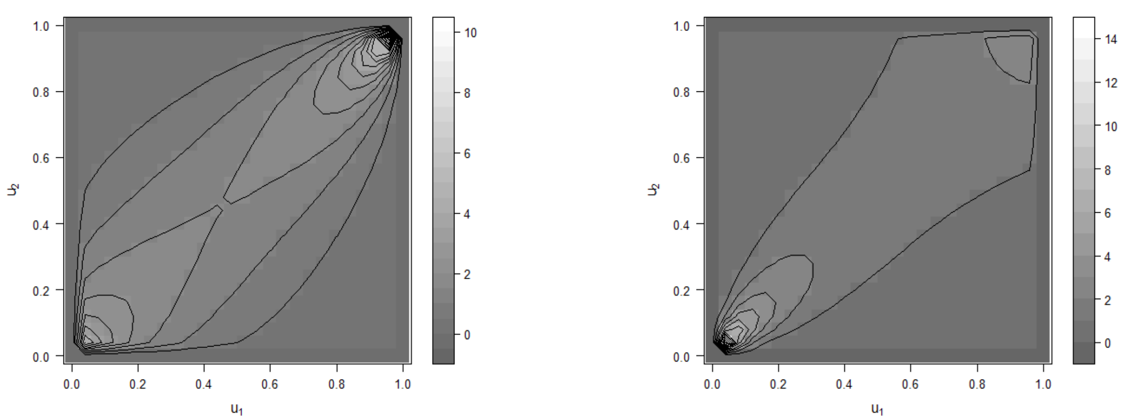


(a) AA on vertical axis **(b)** CP on vertical axis **(c)** TN on vertical axis

Fig. 2: Scatter Plots of Returns



(a) The Density of Gaussian Copula **(b)** The Density of Student t Copula



(c) The Density of Gumbel Copula **(d)** The Density of Clayton Copula

Fig. 3: Density of Four Pair Copulas

sub-figure of Fig. 2). Without a loss of generality, we take three raw materials as examples in this study. They are aluminium alloy (AA), copper (CP) and tin (TN), which have a similar dependence structure with other raw materials traded on the London Metal Exchange. However, the Clayton Archimedean copula is characterised by a single parameter, which reduces the accuracy of predictions as its dimensions increase. This weakness makes the Archimedean Clayton copula unlikely to capture varying degrees of dependence structures among multi-variable vectors. To overcome this limitation, we introduce the vine copula into our research. Compared with multivariate Archimedean Clayton copulas, the vine copula is more flexible since it can simultaneously describe varying degrees of dependence structures through iterative conditioning (Aas et al., 2009). The following details how the multivariate vine copula uses pair-copula functions to capture varying degrees of the dependence structure of variable vectors.

A joint density function $f(x_1, ..., x_n)$ can be decomposed without loss of generality by iteratively conditioning as shown in the following:

$$f(x_1, x_2, ..., x_n) = f_n(x_n) f(x_{n-1}|x_n) f(x_{n-2}|x_{n-1}, x_n)...f(x_1|x_2, ..., x_n) \tag{17}$$

Using conditional copulas, each factor on the right side of Eq.(17) can be decomposed further. For example, when $n = 2$, $f(x_1, x_2) = c_{12}[F_1(x_1), F_2(x_2)] \cdot f_1(x_1) f_2(x_2)$. Using $f(x_1, x_2) = f_2(x_2) f(x_1|x_2)$, we can easily obtain $f(x_1|x_2) = c_{12}[F_1(x_1), F_2(x_2)] \cdot f_1(x_1)$. It is now clear that the second factor, $f(x_{n-1}|x_n)$, on the right side of Eq. (17) can also be decomposed into the pair-copula $c_{(n-1)n}[F_{n-1}(x_{n-1}), F_n(x_n)]$ and a marginal density $f_{n-1}(x_{n-1})$. Similarly, each term in Eq. (17) where ν_j is an arbitrarily chosen component of $\boldsymbol{\nu}$, and $\boldsymbol{\nu}_{-j}$ is vector $\boldsymbol{\nu}$ without this component.

From Eq.(19), we can see that marginal conditional distributions of the form $F(x|v)$ are included in the pair-copula construction. Jose et al. (1996) reveal that for every j:

$$F(x|\boldsymbol{\nu}) = \frac{\partial C_{x,\nu_j|\boldsymbol{\nu}_{-j}}[F(x|\boldsymbol{\nu}_{-j}), F(\nu_j|\boldsymbol{\nu}_{-j})]}{\partial F(\nu_j|\boldsymbol{\nu}_{-j})} \tag{18}$$

can generally be decomposed into the pair-copula multiplied by conditional marginal density. The general formula for an n-dimensional vector $\boldsymbol{\nu}$ is as follows:

$$f(x|\boldsymbol{\nu}) = c_{x\nu_j|\boldsymbol{\nu}_{-j}}[F(x|\boldsymbol{\nu}_{-j}), F(\nu_j|\boldsymbol{\nu}_{-\boldsymbol{j}})] \cdot f(x|\boldsymbol{\nu}_{-\boldsymbol{j}}) \tag{19}$$

where $\boldsymbol{\nu}_{-j}$ is the vector $\boldsymbol{\nu}$ and excludes the component ν_j, and $C_{x,\nu_j|\boldsymbol{\nu}_{-j}}$ is a bivariate copula distribution function. If we assume vector $\boldsymbol{\nu}$ to be one-dimensional, we have

$$F(x|\nu) = \frac{\partial C_{x\nu}[F(x), F(\nu)]}{\partial F(\nu)} \tag{20}$$

When x and ν are uniform (i.e., $f(x) = f(\nu) = 1$, $F(x) = x$ and $F(\nu) = \nu$), the conditional distribution function can be represented by the function $h(x, \nu, \Theta)$, as follows:

$$h(x, \nu, \Theta) = F(x|\nu) = \frac{\partial C_{x\nu}(x, \nu, \Theta)}{\partial \nu} \tag{21}$$

Parameter ν is the conditional variable, and Θ is the parameter for the bivariate copula $C_{x,\nu}(x, \nu)$. In the real-world application, $h(x, \nu, \Theta)$ and the inverse of h-function $h^{-1}(x, \nu, \Theta)$ are iteratively used for sampling and inference for each pair-copula in the vine (the h-function and its inverse of Clayton copula is in Appendix A.2).

Although there are other vine copulas, such as D-vine and regular vine, here, we select the canonical vine due to the efficiency of its hierarchical structure (Aas et al., 2009). If the key variable that governs the interactions in the data set is identified during the modelling process, it can be designated as the root of the canonical vine. Consider as an example the joint density of three-dimensional case $\boldsymbol{X} = (X_1, X_2, X_3)$; here, X_1 can be seen as the root of the canonical vine when X_1 governs X_2, and X_3. To build the most accurate canonical vine, we need to choose the right collateral return series as the root. The root is the collateral that has the highest degree of correlation with the other collaterals. Here, we provide the following formula to find the series that has the highest degree of correlation with all the other series:

$$Z_{x_i} = \sum_{j=1}^{N} |\zeta_{ij}|, \quad \text{where} \quad i, j \in N \tag{22}$$

ζ_{ij} is an $N \times N$ matrix of the Kendall rank correlation coefficient between each pair of collateral units. The collateral that returns X_i with the highest absolute correlation with all the other collateral units is located as the root of the canonical vine. Once the canonical vine copula is chosen, the dependence structure of a portfolio of n collateral units will be parameterised with $\frac{n(n-1)}{2}$ pairwise Clayton copula parameters. In this research, the number of collateral units is three; thus we need to calculate three copula parameters.

Proposition 3. *There exists a PDF $f_{123} = f_1 \cdot f_2 \cdot f_3 \cdot c_{13} \cdot c_{23} \cdot c_{12|3}$ that can be used to simulate the returns of three collateral units. f_n denotes the marginal PDFs and c_n denotes the pairwise copula PDFs (The proof is in Appendix A.3).*

Proposition 3 shows the predictive model that can be used to predict the future returns from collateral. By dynamically predicting future returns for three collateral units, we can iteratively parametrise the function of the probability of loss ($\mathbb{P}(\ln P_i \leq \ln Z)$).

4. Analysis

The analysis section contains three parts that focus on comparing the predictive performance of copulas with that of the MVN, testing the performance of setting a uniform impawn rate with that of setting separate impawn rates, and examining how to determine if willingness to take risks, the liquidity risk of collateral, the interest rate, and the industrial impawn rate affect profit and the settlement of the impawn rate.

4.1. Data

The data set contains monthly collateral returns for three raw materials (i.e. aluminium alloy (AA), copper (CP), and tin (TN)), all of which are common types of collateral in the industrial supply chain. The period for these collateral returns extends from 1/29/1999 to 12/31/2018, yielding a total of 720 observations. The first 360 observations from 1/29/1999 to 12/31/2008 are used for estimating parameters, and the remaining 360 monthly return observations served as an out-of-sample set to test the effectiveness of the Clayton canonical vine copula. To determine a reliable predictive model, we follow DeMiguel et al. (2009), Low et al. (2013) and Sahamkhadam et al. (2018) and use a *'rolling window'* approach to predict future collateral price volatility. In the inventory financing setting, we define the *'rolling window'* approach as *'using the data within previous funding cycles to parameterise the multivariate probability distribution after the* $t = w + 1$ *funding cycle'*. To dynamically estimate the parameters of copulas, we first construct the function (CopulasSimulation3) to estimate copulas. Then, we adopt 'for' syntax to update observations and plunge the updated observations into the function to decide the dependency structure and estimate parameters of copulas. The detailed procedure of adopting a rolling window approach is provided in Appendix A.4.

Based on the historical data, the PDF in Proposition 3 is used to simulate the returns of three collateral units (i.e., $n = 3$). The simulated returns are used to parameterise the function of the default probability for each collateral type. Relevant parameters are set as $\bar{\theta} = 0.7$, $\tau = 0.01$, $r = 1.1\%$, $k = 1$ Month, and $m = 6$ Month. Other parameters are for specific collateral units. For the first collateral (AA), $\alpha_1 = \$500,000$, $\rho_1 = 0.2$, and $g_1 = \$10,000$/Million tons/Month. For the second collateral (CP), $\alpha_2 = \$1000,000$, $\rho_2 = 0.2$, and $g_2 = \$10,000$/Million tons/Month. For the third collateral (TN), $\alpha_3 = \$1500,000$, $\rho_3 = 0.2$, and $g_3 = \$10,000$/Million tons/Month.

4.2. The Predictive Performance of Copulas

Based on the simulated returns and objective function, we iteratively adjust the optimal impawn rate for the collateral in each funding cycle. Because the funding cycle interval is set as six months, there is a total of 20 funding cycles. By using Eq. (22), we find that the cumulative Kendall value of CP is consistently higher than AA and TN thus can be set as the root of Clayton canonical vine copula in all rolling windows. Interestingly, the cumulative Kendall value has an increasing trend in the first 10 rolling windows and then goes down (See Fig. 4). This happens because the positive correlation among three metals is strengthened during the financial crisis. With the window rolling, more observations during the financial crisis are included, and the cumulative value has an increasing trend before the 10^{th} rolling window. When the cumulative value reaches the summit and the window keeps rolling, the observations during the financial crisis are gradually excluded and more observations in the normal period are included, resulting in the decrease of cumulative Kendall value after the 10^{th} rolling window (See the shadow in Fig. 4).

Fig. 4: Cumulative Kendall Value in Each Rolling Window

To justify the effectiveness of the Clayton canonical vine copula for capturing the dependency structure of the time series, we directly use 20 sets of out-of-sample data to calculate the optimal impawn rate [3]. We use the Clayton canonical vine copula and the MVN to dynamically simulate returns on collateral. Based on the simulated returns, we parameterise the function of default probability and calculate the optimal impawn rate. We then calculate the difference between the optimal impawn rate produced by the out-of-sample data and that produced by the Clayton canonical vine copula, and the difference between the optimal impawn rate produced by the out-of-sample data and that produced by the MVN. Based on a comparison of the differences, we can evaluate whether the Clayton canonical vine copula performs better than the classic MVN for parameterising the

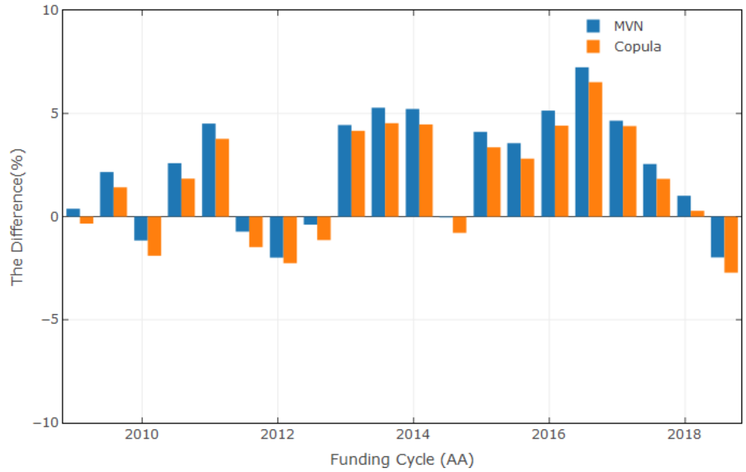
[3] The length of each funding cycle is set as six months in this study. There are 120 months from 1/29/2009 to 12/31/2018; thus there are 20 funding cycles in total.

function of default probability.

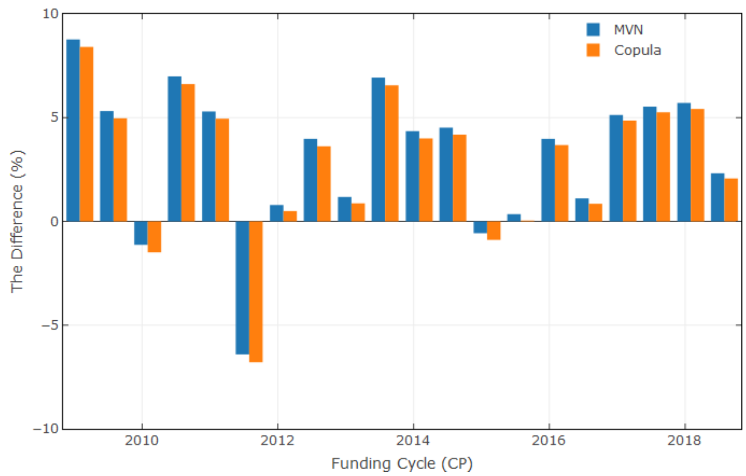
Based on the blue and orange bars in Fig. 5, we can compare the performance of the MVN and copulas. We find that the impawn rate produced by the Clayton canonical vine copula is obviously higher than that produced by the MVN. The reason is that the simulated return produced by the canonical vine copula has a lower variance but higher mean compared with the return produced by the MVN. For the first type of collateral (AA), copulas perform better than the MVN for 14 funding cycles (See Fig. 5a). For the second type of collateral (CP), copulas perform better than the MVN for 17 funding cycles (See Fig. 5b). For the third type of collateral (TN), copulas perform better than the MVN for 15 funding cycles (See Fig. 5c). It is clear that copulas generally perform better than the MVN for most funding cycles, which means the Clayton canonical vine copula has an advantage when capturing the dependency structure and can thus be used to parameterise the function of the default probability. As Hansen et al. (2010) suggested, a more complex and flexible copula with more than one time-varying parameter may be able to more precisely predict future returns. This finding is similar to the conclusions drawn by Brechmann & Czado (2013) and Low et al. (2013). Brechmann & Czado (2013) claimed that highly dimensional vine copulas can accurately capture the characterisation of extreme dependency in the equity and bond markets and can thus be used to more accurately manage financial risk. By setting CVaR as an objective function, Low et al. (2013) concluded that the Clayton canonical vine copula can more accurately set weights for a portfolio than the MVN.

4.3. Optimal Impawn Rate

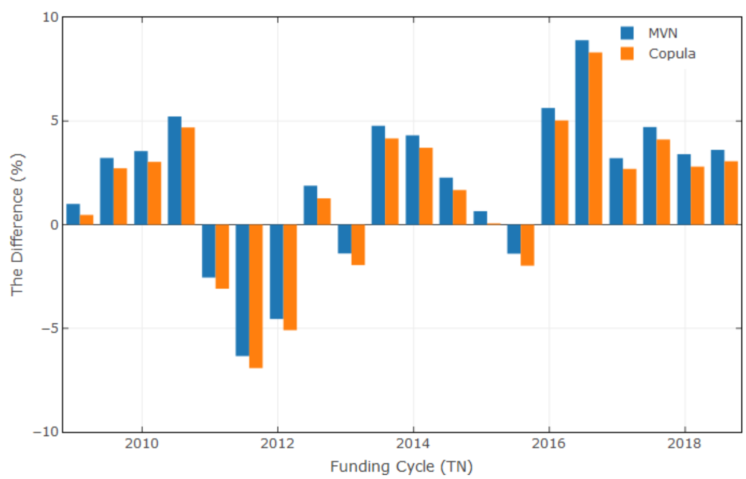
Based on the chosen canonical vine copula-based approach, we calculate the optimal impawn rate for each type of collateral using Eq.(11). To demonstrate the advantage of setting different impawn rates, a uniform impawn rate is also set to maximise the total profit (See Eq. (11)). The optimal impawn rates for AA, CP, and TN are 0.7, 0.68, and 0.69, respectively, in the final funding cycle (from 6/30/2018 to 12/31/2018). The impawn rate for AA is higher than CP and TN because the variance of the simulated returns for AA is lower than for the other two types of collateral. The optimal impawn rate for the combination of the three types of collateral is 0.69. The total profit in the case of separation is $ 179,313.60, and the total profit in the case of combination is $ 179,020.7. Therefore, setting the impawn rate separately for different collateral units can yield higher profit for the inventory financing business. This conclusion is partially supported by He et al. (2012). Based on the historical prices of steel, the authors demonstrated the benefit of setting impawn rates for specific collateral units.



(a) Aluminum Alloy

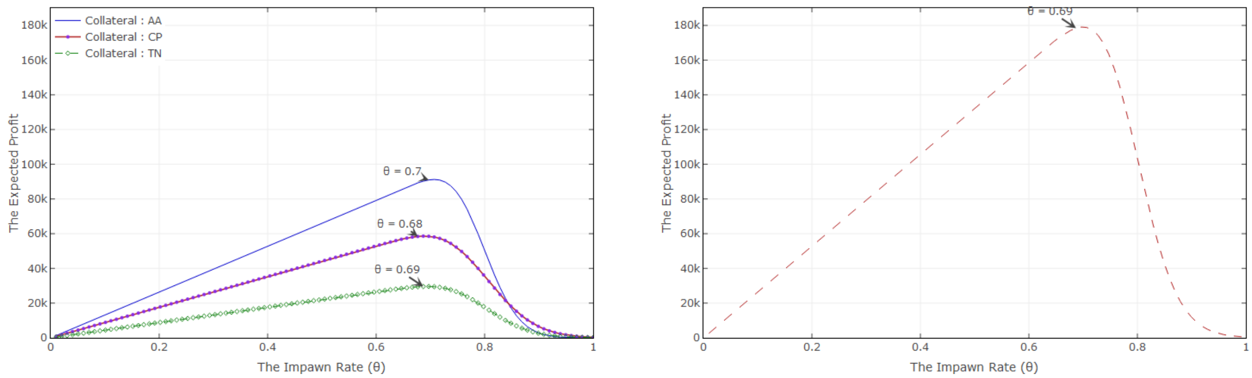


(b) Copper



(c) Tin

Fig. 5: Comparison of the Performance of Copulas and the MVN



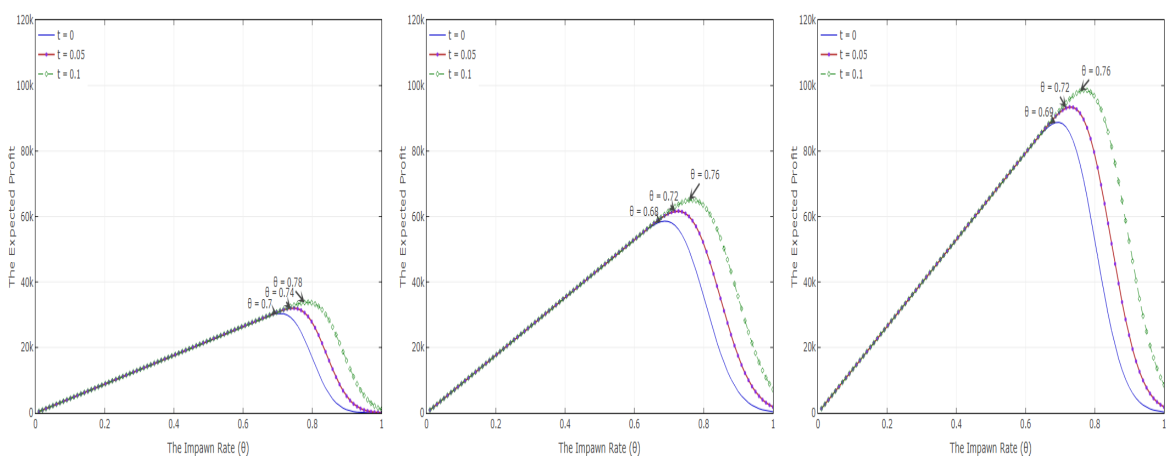
(a) Optimal Impawn Rate for Each Collateral **(b)** Uniform Impawn Rate for All Collateral

Fig. 6: Optimise the Profit of Inventory Financing with Impawn Rates

4.4. Sensitivity Analysis for Different Collaterals

In the following, four sensitivity analyses are provided to help IFPs identify which factors they should focus on when setting optimal impawn rates for different collateral units.

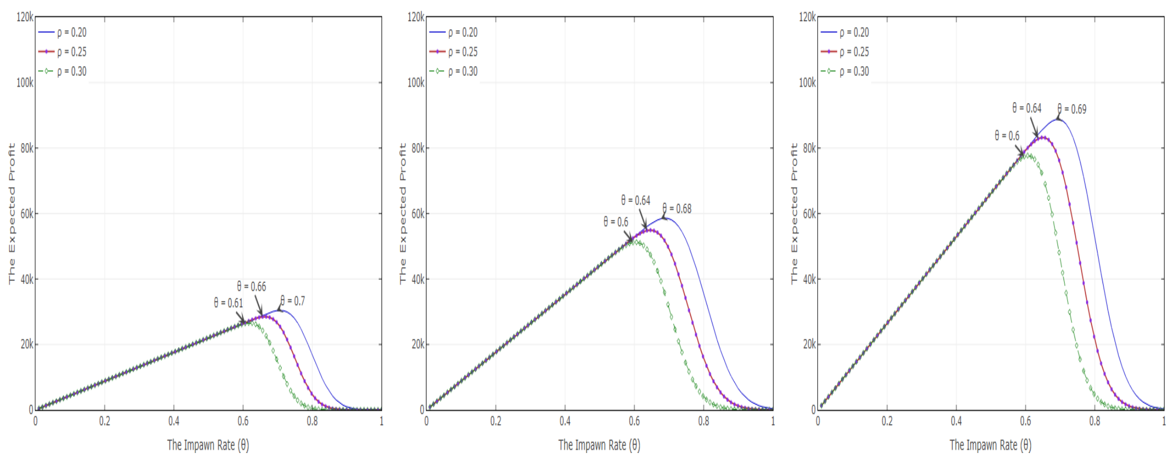
A higher willingness to take risks means the IFP can tolerate more risk that is derived from the default scenario. Fig. 7 shows how the IFP's willingness to take risks influences the optimal impawn rate and the expected profit of the inventory financing business. Based on the analytical results, we find that the optimal impawn rate increases with an increase in the willingness to take risks. Therefore, the IFP should consistently improve her risk-taking ability as this can improve her competitiveness in the inventory financing market (higher optimal impawn rate and expected profit).



(a) AA **(b)** CP **(c)** TN

Fig. 7: The Effect of willingness to Take Risks on the Optimal Impawn Rate

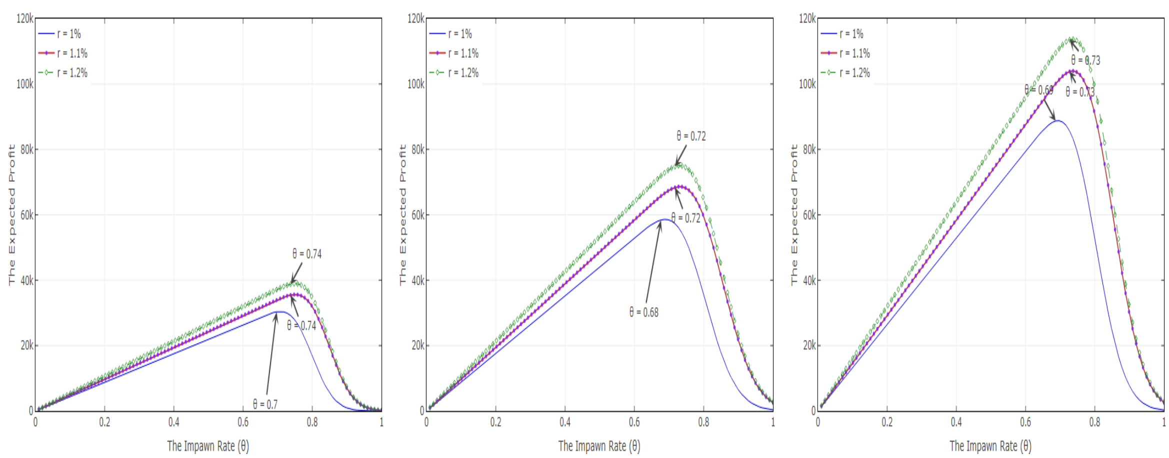
Collateral that is hard to convert into money usually has a high liquidity risk. Based on Fig. 8, we can see that the optimal impawn rates decreases with an increase in liquidity risk. Therefore, the IFP should be very selective regarding collateral. If IFPs want to simultaneously improve their competitiveness and increase profit, they should provide more benefits for borrowers whose collateral has a lower liquidity risk. This finding has been empirically demonstrated by Brunnermeier & Pedersen (2008) and Gorton & Metrick (2012). Brunnermeier & Pedersen (2008) constructed a model that links an asset's liquidity risk and a trader's funding liquidity, showing that an asset's liquidity risk decreases the impawn rate. Gorton & Metrick (2012) claimed that the repo haircut has a strong relationship with confidence in the market. When confidence is low, asset liquidity is low and thus the impawn rate will also be low.



(a) AA **(b)** CP **(c)** TN

Fig. 8: The Effect of Liquidity Risk on the Optimal Impawn Rate

The impawn rate also has a positive relationship with the interest rate but the marginal effect is diminishing. Take AA, for example; when the monthly interest rate increased from 1.0% to 1.1%, the optimal impawn rate increased from 0.70 to 0.74 (See Fig. 9a). Initially, an increase in the interest rate motivates the IFP to set a higher impawn rate as she is willing to take more risk to earn more money. However, when the interest rate increases from 1.1% to 1.2%, there is little effect on the optimal impawn and it is still 0.74, which shows the marginal effect of the interest rate is decreasing. Although an increase of the interest rate can motivate the borrower to increase the impawn rate to attract more funding demand, the IFP will still take financing risks. An overly high impawn rate will result in the IFP losing the utility gained from the increase in the interest rate. This kind of positive relationship between the interest rate and impawn rate has already been identified by the existing literature. For example, Aguiar & Gopinath (2006) and Wu et al. (2017) empirically justified the positive correlation between the default rate and interest rate. Take the repo market as an example; Eren (2014) claimed that over-collateralisation drives down the repo (interest) rate. Within the mechanism of setting the interest rate, the impawn rate can indirectly affect the interest rate by influencing the probability that the borrower defaults (Bakoush et al., 2019) and the level of over-collateralisation (Eren, 2014). Eren (2014) claimed that over-collateralisation is increased by a higher haircut (lower impawn rate). Hence, repos with a higher haircut (lower impawn rate) receive a lower repo (interest) rate. However, none of these studies identify that the positive relationship between the impawn rate and interest rate is marginally diminishing. In addition, based on the analytical result in Fig. 9, we can see that the optimal impawn rate and optimal expected profit increase with an rise in the interest rate. Therefore, in general, if the determined impawn rate does not expose the IFP to more risk, it is easier for her to make more money.

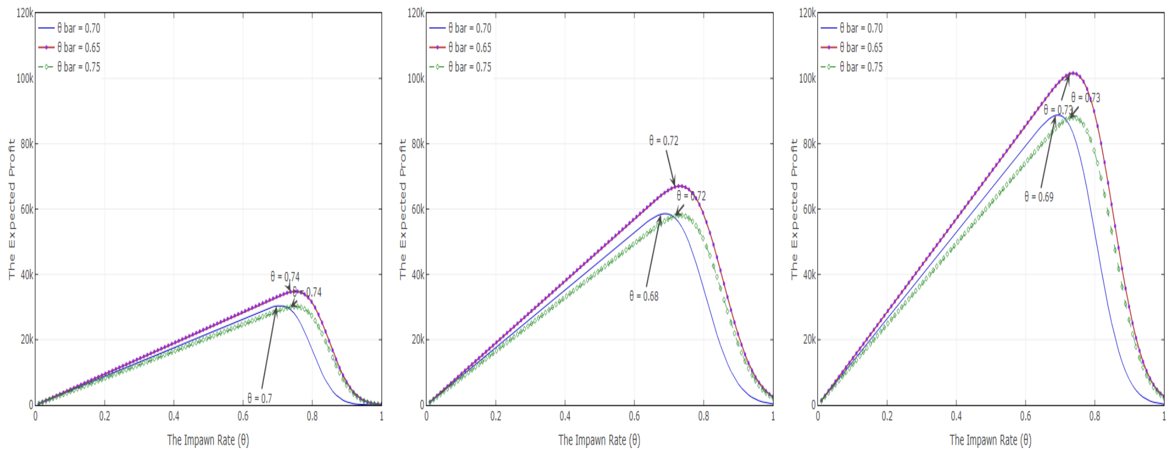


(a) AA **(b)** CP **(c)** TN

Fig. 9: The Effect of the Interest Rate on the Optimal Impawn Rate

The industrial impawn rate is the standard impawn rate that is widely used in the inventory financing market. Different from previous sensitivity analysis, a strong relationship does not exist between the optimal impawn rate and the industrial impawn rate. However, the industrial impawn rate has a significant effect on expected profit. When the industrial impawn rate is low, the IFP can gain more profit from the inventory financing business providing the impawn rate is not set too high or too low. Based on the partial line before the optimal impawn rate in Fig. 10, we find that the expected profit increases with the impawn rate increases. However, there is no such relationship on the partial line after the optimal impawn rate, which means the IFP needs to be especially careful to not set the impawn rate too high when the industrial impawn rate is high, as this will result in more uncertainties for the IFP. This finding generates some insights about the existing literature.

Although the prevailing studies have already investigated the relationship between the impawn rate, liquidity risk, and the interest rate (Aguiar & Gopinath, 2006; Boissel et al., 2017; Luo & Wang, 2018; Wu et al., 2017), the investigation of how the industrial impawn rate influences the decision of individual financial service providers remains vague.



(a) AA **(b)** CP **(c)** TN

Fig. 10: The Effect of the Industrial Impawn Rate on the Optimal Impawn Rate

5. An Extended Model

In practice, the default motivation of borrowers is influenced by not only the fluctuating collateral prices but also the borrowers' capital status. Therefore, when providing inventory financing services to borrowers, it is worthwhile to take the factor of the borrower into consideration, such as evaluating the capital status of each borrower and then calculating the optimal impawn rate for each collateral unit based on the modified function of default probability (See Fig. 11). In short, the extended model takes borrowers' financial status into consideration, along with their collateral portfolios.

The probability that a borrower defaults is independent of the loan. In this case, the IFP would simultaneously consider the capital status of the borrower and the probability of loss on the i^{th} collateral unit; therefore, we can determine the function of joint probability distribution for the borrower on the i^{th} collateral unit:

$$\mathbb{P}^* = [1 - \exp(-\lambda_m)]\mathbb{P}(\ln P_i \leq \ln Z) \tag{23}$$

In Eq. (23), we use the exponential distribution function to measure the default probability of the borrower, which is similar to Kouvelis & Zhao (2012) and Wang et al. (2018). However, in contrast to these works, we also consider the effect of fluctuating collateral prices on default

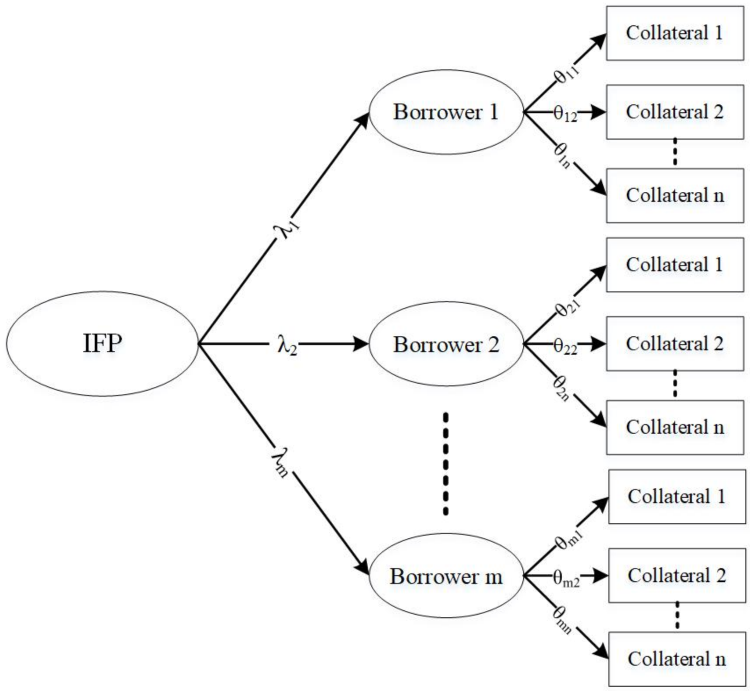


Fig. 11: Optimal Impawn Rate for Individual Borrowers

probability, which can be dynamically parametrised with the Clayton canonical vine copula. Using Eq. (23), we can then further extend Proposition 1 to Proposition 4:

Proposition 4. *When the IFP can tolerate only the loss $\bar{M}_{i,0}$, $\bar{M}_{i,0} = \tau M_{i,0}$ (τ is the level of risk that the IFP can tolerate). For the m^{th} borrower, the probability of loss on the i^{th} collateral is $\mathbb{P}^* = [1 - \exp(-\lambda_m)]\mathbb{P}(\ln P_i \leq \ln Z)$. $P_i = \frac{p_{i,j}}{p_{i,j-1}}$ and $Z = \frac{\theta_i(1-\tau)\exp(kr)}{1-\rho_i}$.*

Proposition 4 also describes the probability of loss on the i^{th} collateral unit, which is affected by both the impawn rate (θ_i) set by the IFP and capital status (λ_m). With an increase in the impawn rate set by the IFP and the deterioration of capital status, the probability of loss would also increase.

Based on Proposition 4, we further design a new profit function for the i^{th} collateral unit of the m^{th} borrower:

$$\pi_i(\theta_i) = \sum_{j=1}^{m}[\alpha_i + \beta_i(\theta_i - \bar{\theta})](1 - \mathbb{P}^*)[\exp(kjr) - \exp(k(j-1)r)] - G_i \tag{24}$$

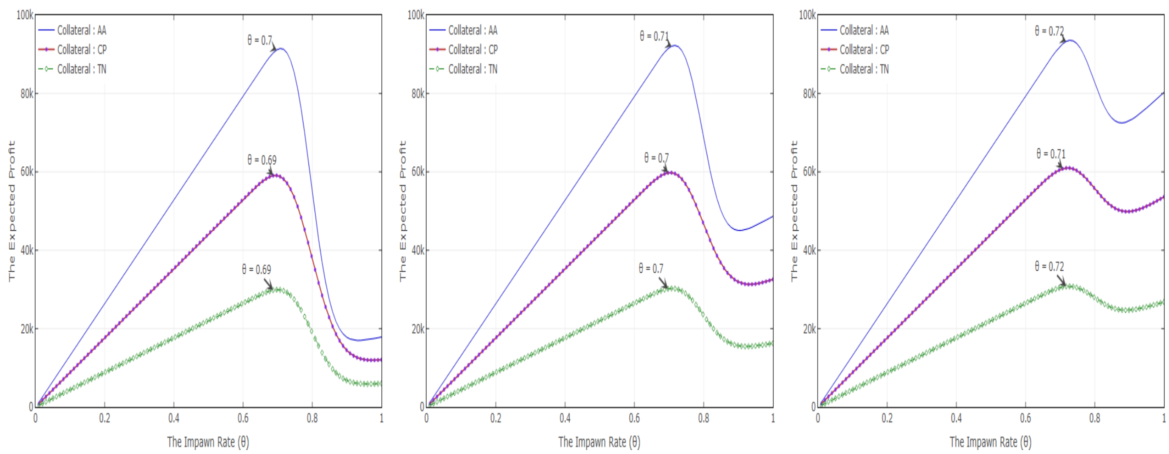
$(1 - \mathbb{P}^*)$ is the probability that the borrower who owns the i^{th} collateral unit does not default. Therefore, the new expected revenue for the i^{th} collateral unit in the j^{th} interval is $[\alpha_i + \beta_i(\theta_i - \bar{\theta})](1 - \mathbb{P}^*)[\exp(kjr) - \exp(k(j-1)r)]$. Eq. (24) can be further simplified into:

$$\pi_i(\theta_i) = [\alpha_i + \beta_i(\theta_i - \bar{\theta})](1 - \mathbb{P}^*)[\exp(kjr) - 1] - G_i \tag{25}$$

To calculate the optimal impawn rate with Eq. (25), we can take its first-order and second-order derivative. Here, we have Proposition 5.

Proposition 5. *When $\bar{F}(\ln\theta_i + \omega)$ interacts with $f(\ln\theta_i + \omega) + (1 - \frac{1}{1-\exp(-\lambda_m)})$, there is a single optimal impawn rate θ_i^* in the interval $(0, \min(\exp(\delta + \mu - \omega), 1))$ that maximises the profit of the IFP on the i^{th} collateral unit, and $\omega = \ln\frac{1-\tau}{1-\rho_i} + kr$ (The proof is in Appendix A.5).*

In Proposition 5, μ and δ are the mean and variance of simulated collateral returns. From the proof in Appendix A.5, we know that when $\bar{F}(\ln\theta_i+\omega)$ interacts with $f(\ln\theta_i+\omega)+(1-\frac{1}{1-\exp(-\lambda_m)})$, there is an optimal impawn rate in the interval $(0, \min(\exp(\delta+\mu-\omega), 1))$ that maximises the profit of the IFP on the i^{th} collateral unit. Otherwise, the profit function tends to be linear. Based on the calculated optimal impawn rate for each collateral unit, we can calculate the total optimal profit of the inventory financing business.



(a) The First Borrower **(b)** The Second Borrower **(c)** The Third Borrower

Fig. 12: The Optimal Impawn Rate for Different Borrowers with the Same Collateral

Based on Eq. (25), we can calculate the optimal impawn rate for borrowers who have a different capital status. Here, we have three borrowers, each of whom has his own capital status, which can be revealed by the default motivation (λ_m). Good capital status means low default motivation, and bad capital status means high default motivation. In the section on the sensitivity analysis, the default motivations are separately set as 2, 1, and 0.5. The analytical results show that the optimal impawn rate decreases as the capital status of the borrower deteriorates. For the first borrower, the separate optimal impawn rates of AA, CP, and TN are 0.7, 0.69, and 0.69, respectively. For the second borrower, whose default motivation is 1, the separate optimal impawn rates for the three kinds of collateral are 0.71, 0.7, and 0.7, respectively. For the third borrower, who has the lowest default motivation, the separate optimal impawn rates for the three kinds of collateral further increase to 0.72, 0.71, and 0.72, respectively. Therefore, borrowers who have low default motivation usually have relatively low risk and thus as higher impawn rate can be set (See Fig. 12). Interestingly, on the right side of Fig. 12, the expected profit increases as the default motivation of the borrower decreases. Further, on the right side of Fig. 12b and Fig. 12c, there is an impawn rate that minimises the expected profit. Therefore, when considering the factor of the borrower, there is a threshold. An impawn rate above that threshold can produce lower expected profit than the profit produced by the highest impawn rate that an IFP can set ($\theta_i = 1$). However, this does make sense. When a borrower has abundant capital ($\mathbb{P}^* \approx 0$), the function of expected profit would be linear (i.e., $\pi(\theta_i) = [\alpha_i + \beta_i(\theta_i - \bar{\theta})][\exp(kjr) - 1] - G_i$. In this case, to maximise the expected profit, the optimal impawn rate for this kind of borrower should be set as 1.

6. Conclusion

This research introduces impawn rate optimisation into inventory financing from an IFP perspective, thereby helping IFPs improve their competitiveness in the inventory financing business. First, we compare the predictive performance of MVN and copulas, justifying the advantage of copulas for simulating future returns of collateral. Furthermore, we compare the expected profit produced by a uniform impawn rate and multi-impawn rates, showing that setting multi-impawn rates can help IFPs gain more profit. Based on the evaluation, a sensitivity analysis is provided to illustrate which factors IFPs should consider when setting multi-optimal impawn rates for different types of collateral. Finally, we extend the model by providing inventory financing service to borrowers who have different default motivations. By considering the factor of the borrower, IFPs can further differentiate their inventory financing business by providing customised impawn rates. Specifically, the main research findings are as follows:

By comparing the predictive performance of the MVN and the Clayton canonical vine copula, we find that the latter can evaluate default probability very well and thus be used to help the IFP parameterise the objective profit function of the inventory financing business. Although the existing literature seldom investigates the Clayton canonical vine copula's ability to manage default risk, its ability to predict future returns more precisely has already been underlined (Brechmann & Czado, 2013; Low et al., 2013). Furthermore, based on the chosen copula, we calculate and compare the expected profit with a uniform impawn rate and multiple impawn rates. We find that setting multiple impawn rates can help IFPs gain more profit and thus be applied in the inventory financing business. This analytical result is intuitive; the settlement of multiple optimal impawn rates is based on the optimisation of each objective profit function. However, the settlement of a single optimal impawn rate is based on the sum of three objective profit functions, which induces triple marginalisation (Chen et al., 2018). Similarly, Buzacott & Zhang (2004) investigated performance when setting heterogeneous interest rates. They demonstrated that financial service providers can optimise asset-based financing by choosing an appropriate interest rate for each borrower. To further take advantage of the Clayton canonical vine copula, which characterises the dependency structure among time series very well, the original business model can be extended to another business model, namely, setting the optimal rate based on borrowers who have different capital status. Based on the analysis, we find that for borrowers with low default motivation, a higher optimal impawn rate can be set, even considering the effect of fluctuating collateral prices on default probability. With a decrease in default motivation, the expected profit function tends to be linear. In extreme cases, if the borrower has very good credit (the value of λ_m causes $\bar{F}(\ln\theta_i + \omega)$ to not interact with $f(\ln\theta_i + \omega) + (1 - \frac{1}{1-\exp(-\lambda_m)})$, the optimal impawn rate can be set as 1. Thus, when providing inventory financing, IFPs, especially, need to look at the historical credit of borrowers. Based on the evaluated default motivation of borrowers, they can set more accurate impawn rates.

This research has significant managerial implications. First, we derive the optimal impawn rate for inventory financing, which is beneficial for IFPs engaged in the inventory financing business. We suggest that IFPs can actively deal with the fluctuation of collateral prices via the settlement of the impawn rate in different funding cycles. Second, the factors identified in the sensitivity analysis can also provide IFPs with some insights on how to set the impawn rate. For example, when an IFP sets a high interest rate, it is necessary for her to take special care to not set an overly high impawn rate as this can increase the default probability and drag down the profit that the IFP can gain from the inventory financing business. When financial providers in the inventory financing market are all risk averse and inclined to set a low impawn rate, IFPs can set a higher impawn rate (if the default probability function suggests the IFP should do so), thereby providing the IFP with a competitive advantage. Third, the model provided by this research is very flexible, which means it can also be introduced in the financial field to optimise the financing of pawned and fund stocks.

This study is the first attempt to adopt canonical vine copulas to investigate how IFPs can set the optimal impawn rate for the inventory financing business. This research can be extended in three ways. First, future research can construct the relationship between the interest rate and the impawn rate, thereby setting both an optimal impawn rate and interest rate for the inventory financing business. In this research, the interest rate is assumed to be exogenous. In actuality, the interest rate is also strongly linked with the impawn rate (Aguiar & Gopinath, 2006; Eren, 2014). When the impawn rate is low, the IFP undertakes less risk, which means she is willing to provide a discounted interest rate to her borrowers to attract more financing. Second, the current development of copulas is rapid (Fan & Patton., 2014), and this research selects the Clayton Copula, the most classical in the copula family. Therefore, future studies could adopt other copulas that have a lower tail and compare their performance with the Clayton canonical vine copula to determine better copulas that can be used to optimise the impawn rate. The final way that the research could be extended is linking copulas with a game theory model. Game theory mainly investigates how two or more entities gamble with each other and finally achieve equilibrium (Wang et al., 2018; Chen & Cai, 2011). The incorporation of copulas can help the entity in the game system evaluate the probability that the other entity will adopt certain strategies, which can increase the preciseness of analysis.

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Appendix A. Proof of Propositions and Pair-copula

Appendix A.1. Proof of Proposition 2

Based on Eq. (11), taking the first-order and the second-order derivative of $\pi_i(\theta_i)$ with the respect to θ_i, we have

$$\frac{d\pi_i}{d\theta_i} = [\exp(kjr) - 1]\beta_i[1 - F(\ln\theta_i + \omega) - f(\ln\theta_i + \omega)] \tag{A.1}$$

and

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i}[\exp(kjr) - 1][f(\ln\theta_i + \omega) + f'(\ln\theta_i + \omega))] \tag{A.2}$$

$\omega = \ln\frac{1-\tau}{1-\rho_i} + kr$. Because the $F(x)$ and $f(x)$ are the CDF and PDF of normal distribution, by using the transformation $F(x) = \Phi(\frac{x-\mu}{\sigma}) = \int_{-\infty}^{x}\phi(\frac{t-\mu}{\sigma})\,dt$ and $f(x) = \phi(\frac{x-\mu}{\sigma})$ ($\Phi(\cdot)$ and $\phi(\cdot)$ are CDF and PDF of standard normal distribution. μ and σ are mean and variance of logarithmic returns of i^{th} collateral unit. Then we further have:

$$\frac{d\pi_i}{d\theta_i} = \beta_i[\exp(kjr) - 1][1 - \Phi(\frac{\ln\theta_i + \omega - \mu}{\sigma}) - \phi(\frac{\ln\theta_i + \omega - \mu}{\sigma})] \tag{A.3}$$

and

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i}[\exp(kjr) - 1][\phi(\frac{\ln\theta_i + \omega - \mu}{\sigma}) + \phi'(\frac{\ln\theta_i + \omega - \mu}{\sigma})] \tag{A.4}$$

Because $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ and $\phi'(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})(-x)$, Eq. (A.4) can be further wrote into

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i\sqrt{2\pi}}[\exp(kjr) - 1]\exp(-\frac{(\frac{\ln\theta_i+\omega-\mu}{\sigma})^2}{2})(1 - \frac{\ln\theta_i + \omega - \mu}{\sigma}) \tag{A.5}$$

If $\frac{d^2\pi_i}{d\theta_i^2} < 0$, then $0 < \theta_i < \exp(\sigma + \mu - \omega)$. Because $0 < \theta_i < 1$, therefore, here we have two cases:
(i) When $\sigma + \mu - \omega < 0$, $0 < \theta_i < \exp(\sigma + \mu - \omega)$.

$$\frac{d\pi_i}{d\theta_i}(\exp(\sigma + \mu - \omega)) = [\exp(kjr) - 1]\beta_i[\bar{\Phi}(1) - \phi(1)] \tag{A.6}$$

Because $\frac{\ln\theta_i+\omega-\mu}{\sigma}$ is monotonically increasing for θ_i, there exists θ_i to make $\frac{d\pi_i}{d\theta_i} = 0$ (See Fig. A.13). Therefore, there exists a single optimal impawn rate θ_i^* in the interval $(0, \exp(\sigma + \mu - \omega))$ to maximise π_i.

(ii) When $\sigma + \mu - \omega \geq 0$, $0 < \theta_i < 1$.

$$\frac{d\pi_i}{d\theta_i}(1) = [\exp(kjr) - 1]\beta_i[\bar{\Phi}(\frac{\omega - \mu}{\sigma}) - \phi(\frac{\omega - \mu}{\sigma})] \tag{A.7}$$

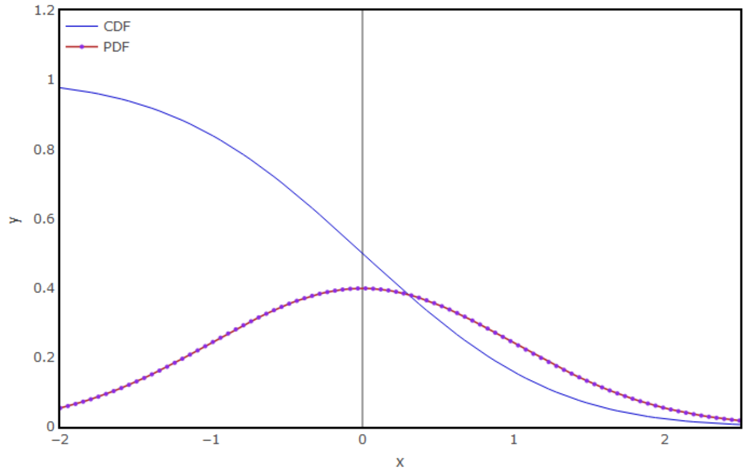


Fig. A.13: The Relationship between the Standard Normal Distribution and Its CDF

Because $\frac{\omega-\mu}{\sigma} > 1$, thus there exists a single optimal impawn rate θ_i^* in the interval of $(0, 1)$ to maximise π_i (See Fig. A.13).

In summary, there exists a single optimal impawn rate θ_i^* in the interval $(0, min(\exp(\sigma + \mu - \omega), 1))$ to maximise π_i.

Appendix A.2. The bivariate Clayton copula

The density of the bivariate Clayton copula is as follows:

$$c(u_1, u_2) = (1 + \delta_{12})(u_1 u_2)^{-1-\delta_{12}} \times (u_1^{\delta_{12}} + u_2^{-\delta_{12}} - 1)^{-1/\delta_{12}-2} \tag{A.8}$$

where $0 < \delta_{12} < \infty$ controls the dependence. $\delta_{12} \to \infty$ implies perfect independence, while $\delta_{12} \to 0$ means independence.

$$h(u_1, u_2, \delta_{12}) = u_2^{-\delta_{12}-1}(u_1^{-\delta_{12}} + u_2^{-\delta_{12}} - 1)^{-1-1/\delta_{12}} \tag{A.9}$$

and the inverse of the h-function is shown as:

$$h_{12}^{-1}(u_1, u_2, \delta_{12}) = \left\{(u_1 \cdot u_2^{\delta_{12}+1})^{-\frac{\delta_{12}}{\delta_{12}+1}} + 1 - \nu^{-\delta_{12}}\right\}^{-1/\delta_{12}} \tag{A.10}$$

Appendix A.3. Proof of Proposition 3

When $n = 3$, the joint density of the three-dimensional case, $f(x_1, x_2, x_3)$, can be represented by a function of the bivariate condition copulas as follows:

$$f(x_1, x_2, x_3) = f_1(x_1) \cdot f(x_2|x_1) \cdot f(x_3|x_1, x_2) \tag{A.11}$$

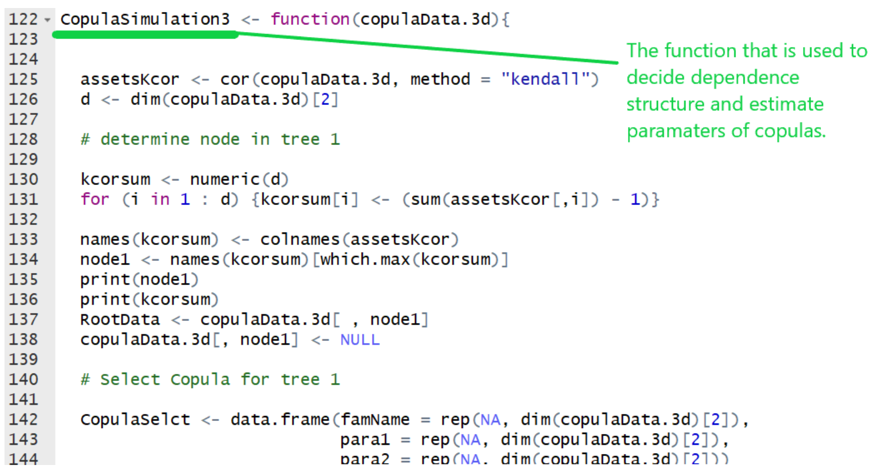
The second factor in the right side of Eq. (A.11) can be decomposed into the pair-copula and a marginal density. We have

$$f(x_2|x_1) = c_{12}[F_1(x_1), F_2(x_2)] \cdot f_2(x_2) \tag{A.12}$$

The third factor in the right side of Eq. (A.12) can be decomposed into

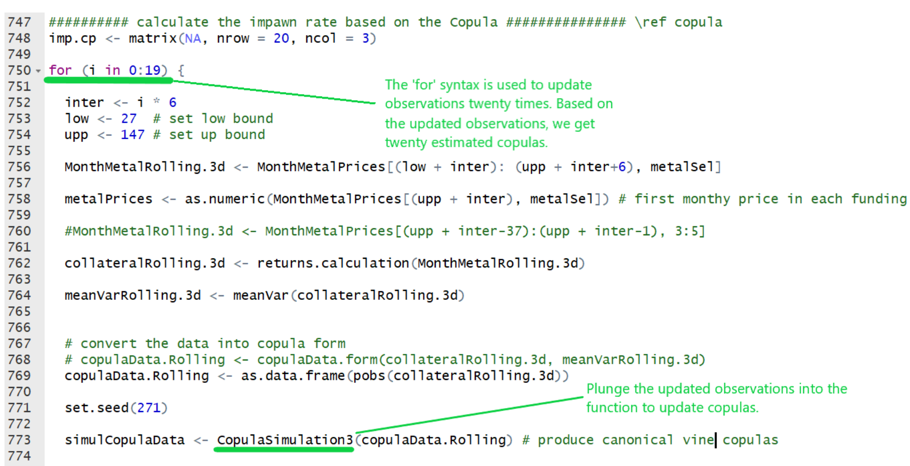
$$\begin{aligned} f(x_3|x_1,x_2) = \frac{f(x_2,x_3|x_1)}{f(x_2|x_1)} &= \frac{c_{23|1}[F(x_2|x_1), F(x_3|x_1)] \cdot f(x_2|x_1) \cdot f(x_3|x_1)}{f(x_2|x_1)} \\ &= c_{23|1}[F(x_2|x_1), F(x_3|x_1)] \cdot f(x_3|x_1) \\ &= c_{23|1}[F(x_2|x_1), F(x_3|x_1)] \cdot c_{13}[F_1(x_1), F_3(x_3)] \cdot f_3(x_3) \end{aligned} \tag{A.13}$$

Appendix A.4. The Procedure of Adopting Rolling Window Approach



```
CopulaSimulation3 <- function(copulaData.3d){


  assetsKcor <- cor(copulaData.3d, method = "kendall")
  d <- dim(copulaData.3d)[2]

  # determine node in tree 1

  kcorsum <- numeric(d)
  for (i in 1 : d) {kcorsum[i] <- (sum(assetsKcor[,i]) - 1)}

  names(kcorsum) <- colnames(assetsKcor)
  node1 <- names(kcorsum)[which.max(kcorsum)]
  print(node1)
  print(kcorsum)
  RootData <- copulaData.3d[ , node1]
  copulaData.3d[, node1] <- NULL

  # Select Copula for tree 1

  CopulaSelct <- data.frame(famName = rep(NA, dim(copulaData.3d)[2]),
                            para1 = rep(NA, dim(copulaData.3d)[2]),
                            para2 = rep(NA, dim(copulaData.3d)[2]))
```


Fig. A.14: The Function Used to Construct Copulas



```
########## calculate the impawn rate based on the Copula ############### \ref copula
imp.cp <- matrix(NA, nrow = 20, ncol = 3)

for (i in 0:19) {

  inter <- i * 6
  low <- 27  # set low bound
  upp <- 147 # set up bound

  MonthMetalRolling.3d <- MonthMetalPrices[(low + inter): (upp + inter+6), metalSel]

  metalPrices <- as.numeric(MonthMetalPrices[(upp + inter), metalSel]) # first monthy price in each funding

  #MonthMetalRolling.3d <- MonthMetalPrices[(upp + inter-37):(upp + inter-1), 3:5]

  collateralRolling.3d <- returns.calculation(MonthMetalRolling.3d)

  meanVarRolling.3d <- meanVar(collateralRolling.3d)


  # convert the data into copula form
  # copulaData.Rolling <- copulaData.form(collateralRolling.3d, meanVarRolling.3d)
  copulaData.Rolling <- as.data.frame(pobs(collateralRolling.3d))

  set.seed(271)

  simulCopulaData <- CopulaSimulation3(copulaData.Rolling) # produce canonical vine copulas
```


Fig. A.15: ‘for’ Syntax

Appendix A.5. Proof of Proposition 5

Based on Eq. (25), taking the first-order and the second-order derivative of $\pi_i(\theta_i)$ with the respect to θ_i, we have

$$\frac{d\pi_i}{d\theta_i} = [\exp(kjr) - 1]\beta_i\{1 - [1 - \exp(-\lambda_m)]F(\ln\theta_i + \omega) - [1 - \exp(-\lambda_m)]f(\ln\theta_i + \omega)\} \quad (A.14)$$

and

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i}[\exp(kjr) - 1][1 - \exp(-\lambda_m)][f(\ln\theta_i + \omega) + f'(\ln\theta_i + \omega)] \quad (A.15)$$

$\omega = \ln\frac{1-\tau}{1-\rho_i} + kr$. Because the $F(x)$ and $f(x)$ are the CDF and PDF of normal distribution, by using the transformation $F(x) = \Phi(\frac{x-\mu}{\sigma}) = \int_{-\infty}^{x}\phi(\frac{t-\mu}{\sigma})\,dt$ and $f(x) = \phi(\frac{x-\mu}{\sigma})$ ($\Phi(\cdot)$ and $\phi(\cdot)$ are CDF and PDF of standard normal distribution), then we further have:

$$\frac{d\pi_i}{d\theta_i} = \beta_i[\exp(kjr) - 1]\{1 - [1 - \exp(-\lambda_m)]\Phi(\frac{\ln\theta_i + \omega - \mu}{\sigma}) - [1 - \exp(-\lambda_m)]\phi(\frac{\ln\theta_i + \omega - \mu}{\sigma})\} \quad (A.16)$$

and

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i}[\exp(kjr) - 1][1 - \exp(-\lambda_m)][\phi(\frac{\ln\theta_i + \omega - \mu}{\sigma}) + \phi'(\frac{\ln\theta_i + \omega - \mu}{\sigma})] \quad (A.17)$$

Because $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ and $\phi'(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})(-x)$, Eq. (A.4) can be further wrote into

$$\frac{d^2\pi_i}{d\theta_i^2} = -\frac{\beta_i}{\theta_i\sqrt{2\pi}}[\exp(kjr) - 1][1 - \exp(-\lambda_m)]\exp(-\frac{(\frac{\ln\theta_i + \omega - \mu}{\sigma})^2}{2})(1 - \frac{\ln\theta_i + \omega - \mu}{\sigma}) \quad (A.18)$$

If $\frac{d^2\pi}{d\theta_i^2} < 0$, then $0 < \theta_i < \exp(\sigma + \mu - \omega)$. Because $0 < \theta_i < 1$, thus here we have two cases:

(i) When $\sigma + \mu - \omega < 0$, $0 < \theta_i < \exp(\sigma + \mu - \omega)$.

$$\frac{d\pi_i}{d\theta_i}(\exp(\sigma + \mu - \omega)) = [\exp(kjr) - 1]\beta_i[1 - \exp(-\lambda_m)][\bar{\Phi}(1) - (\phi(1) - \frac{1}{1 - \exp(-\lambda_m)} + 1)] \quad (A.19)$$

Because $\frac{\ln\theta_i + \omega - \mu}{\sigma}$ is monotonically increasing for θ_i, when $\bar{F}(\ln\theta_i + \omega)$ interacts with $f(\ln\theta_i + \omega) + (1 - \frac{1}{1-\exp(-\lambda_m)})$, there exists θ_i to make $\frac{d\pi_i}{d\theta_i} = 0$. Therefore, in this case there exists a single optimal impawn rate θ_i^* in the interval $(0, \exp(\sigma + \mu - \omega))$ to maximise π_i (See Fig. A.13).

(ii) When $\sigma + \mu - \omega \geq 0$, $0 < \theta_i < 1$.

$$\frac{d\pi_i}{d\theta_i}(1) = [\exp(kjr) - 1]\beta_i[1 - \exp(-\lambda_m)]\{\bar{\Phi}(\frac{\omega - \mu}{\sigma}) - [\phi(\frac{\omega - \mu}{\sigma}) - \frac{1}{1 - \exp(-\lambda_m)} + 1]\} \quad (A.20)$$

Because $\frac{\omega-\mu}{\sigma} > 1$, therefore, when $\bar{F}(\ln\theta_i + \omega)$ interacts with $f(\ln\theta_i + \omega) + (1 - \frac{1}{1-\exp(-\lambda_m)})$, there exists a single optimal impawn rate θ_i^* in the interval of $(0, 1)$ to maximise π_i (See Fig. A.13).

In summary, when $\bar{F}(\ln\theta_i + \omega)$ interacts with $f(\ln\theta_i + \omega) + (1 - \frac{1}{1-\exp(-\lambda_m)})$, there exists a single optimal impawn rate θ_i^* in the interval $(0, min(\exp(\sigma + \mu - \omega), 1))$ to maximise π_i.